Rees Algebras of Conormal Modules

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February 1, 2008

Abstract

We deal with classes of prime ideals whose associated graded ring is isomorphic to the Rees algebra of the conormal module in order to describe the divisor class group of the Rees algebra and to examine the normality of the conormal module.

Keywords: Rees algebra, conormal module, divisor class group, integral closure, Cohen–Macaulay ring.

1 Introduction

We investigate several algebras associated to a prime ideal of a commutative Noetherian ring and study the relationships among them by examining normality conditions (divisor class groups, the construction of integral closures, etc). Throughout this paper, let R be a commutative Noetherian ring and $\mathfrak{p}$ a prime ideal of R. We illustrate the algebras to be examined with the following diagram (unexplained terminology will be discussed in the text):

$$\begin{tikzcd}
 & \mathcal{R}(\mathfrak{p}) \arrow[r] & \mathrm{gr}_{\mathfrak{p}}R \arrow[r] \arrow[dd, dotted, no head] \arrow[ddr] & (\mathrm{gr}_{\mathfrak{p}}R)_{\mathrm{red}} \arrow[ddl, dotted, no head] \\
S(\mathfrak{p}) \arrow[ur] \arrow[dr] & & & \\
 & S(\mathfrak{p}/\mathfrak{p}^2) \arrow[r] \arrow[uur] & \boxed{\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)} \arrow[r, dotted, no head] & \mathrm{gr}_{\mathfrak{p}}R/T
\end{tikzcd}$$

where $S(\mathfrak{p})$ is the symmetric algebra of $\mathfrak{p}$, $S(\mathfrak{p}/\mathfrak{p}^2)$ is the symmetric algebra of $\mathfrak{p}/\mathfrak{p}^2$ as an $R/\mathfrak{p}$–module, $\mathcal{R}(\mathfrak{p})$ is the Rees algebra of $\mathfrak{p}$, $\mathrm{gr}_{\mathfrak{p}}R$ is the associated graded ring, $(\mathrm{gr}_{\mathfrak{p}}R)_{\mathrm{red}}$

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Mathematics Subject Classification 2000: 13H10, 13B21, 13C20.
This paper is based on the Ph.D. dissertation of the author at Rutgers University, written under the direction of Professor Wolmer V. Vasconcelos.

is the reduced ring of the associated graded ring, $(\mathrm{gr}_{\mathfrak{p}} R)/T$ is the associated graded ring mod its torsion T, and $\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)$ is the Rees algebra of $\mathfrak{p}/\mathfrak{p}^2$, which is the focus of our study.

Let us recall the notion of the Rees algebra of a module. Let R be a domain and E a finitely generated torsionfree R–module of rank e with an embedding $\varphi : E \hookrightarrow R^e$. The *Rees algebra* $\mathcal{R}(E)$ of E is the subalgebra of the polynomial ring $R[T_1, \ldots, T_e]$ generated by all linear forms $a_1T_1 + \cdots + a_eT_e$, where $(a_1, \ldots, a_e)$ is the image of an element of E in R^e under the embedding. The Rees algebra $\mathcal{R}(E)$ is a standard graded algebra $\bigoplus_{n=0}^{\infty} E_n$ over R with $E_1 = E$. We refer the reader to [4] for a general discussion of Rees algebras of modules over general rings, including the fact that the Rees algebra $\mathcal{R}(E)$ is independent of the embedding φ when R is a domain. In general, there is a surjection from the symmetric algebra of E onto the Rees algebra of E and the module E is said to be of *linear type* if this surjection is an isomorphism. Let $U \hookrightarrow R^e$ be a submodule of E with the same rank as that of E. The module E is *integral* over the module U if the Rees algebra of E is integral over the R-subalgebra generated by U. In this case we say that U is a *reduction* of E. The *integral closure* $\overline{E}$ of E is the largest submodule of R^e which is integral over the module E. If E is equal to $\overline{E}$, then E is called *integrally closed* or *complete*. If the Rees algebra $\mathcal{R}(E)$ of E is integrally closed, then the module E is said to be *normal*. This means that each component E_n of $\mathcal{R}(E)$ is integrally closed (Proposition 4.1).

We focus on the normality of a conormal module $\mathfrak{p}/\mathfrak{p}^2$ as an $R/\mathfrak{p}$–module. In general, the Rees algebra $\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)$ depends only on the module $\mathfrak{p}/\mathfrak{p}^2$ over the ring $R/\mathfrak{p}$, not on the ring R itself. To ensure R has a role, we must force a relationship between $\mathfrak{p}$ and R, which is the case when we assume, for example, that the prime ideal $\mathfrak{p}$ has finite projective dimension over R. Throughout this paper, we denote the associated graded ring of a prime ideal $\mathfrak{p}$ by G, i.e.,

$$\mathrm{G} = \mathrm{gr}_{\mathfrak{p}} R = \bigoplus_{t=0}^{\infty} \mathfrak{p}^t/\mathfrak{p}^{t+1} = \bigoplus_{t=0}^{\infty} \mathrm{G}_t.$$

Our goal is to describe the divisor class group of an integrally closed associated graded ring and to examine the normality of the conormal module whose Rees algebra is isomorphic to the associated graded ring. First, when the associated graded ring is integrally closed, we obtain the following theorem.

Theorem 3.1 *Let R be a Cohen–Macaulay ring and $\mathfrak{p}$ a prime ideal of finite projective dimension. If $R/\mathfrak{p}$ is an integrally closed domain of dimension greater than or equal to 2 and the associated graded ring G of $\mathfrak{p}$ is integrally closed, then the mapping of divisor class groups*

$$\mathsf{Cl}(R/\mathfrak{p}) \ni [L] \mapsto [L\mathrm{G}] \in \mathsf{Cl}(\mathrm{G})$$

is a group isomorphism. In particular, if G is integrally closed and $R/\mathfrak{p}$ is a factorial domain, then G is a factorial domain.

The exact sequence of the components of the embedding $\mathrm{G} \subset \overline{\mathrm{G}}$,

$$0 \to \mathrm{G}_t \longrightarrow \overline{\mathrm{G}}_t \longrightarrow C_t \to 0, \quad t \geq 1,$$

raises the question of when the normality of the associated graded ring G can be detected in the vanishing of C_t for low t. Thus, we study the relationship between the normality of

the conormal module and the completeness of components of the associated graded ring. So far we prove the following.

Theorem 4.3 *Let R be a Gorenstein local ring and $\mathfrak{p}$ a prime ideal generated by a strongly Cohen–Macaulay d–sequence. Suppose that $\mathfrak{p}$ has finite projective dimension, that $R/\mathfrak{p}$ is an integrally closed domain of dimension 2 and that the associated graded ring G of $\mathfrak{p}$ is a domain. Then the conormal module $\mathfrak{p}/\mathfrak{p}^2$ is integrally closed if and only if G is normal.*

This paper is organized as follows. In Section 2, we give a suitable condition for an associated graded ring to be a domain or an integrally closed domain. Our focus is on classes of ideals whose associated approximation complexes are acyclic. In Section 3, we prove our main result regarding the relationship between the divisor class group $\mathsf{Cl}(\mathrm{G})$ of the associated graded ring G and that $\mathsf{Cl}(R/\mathfrak{p})$ of $R/\mathfrak{p}$. In Section 4, we deal with special kinds of prime ideals whose associated graded ring G is isomorphic to the Rees algebra of the conormal module associated to the prime ideal. We also study the normality of G in terms of the completeness of the components of G. More precisely, we would like obtain any statement of the form $\mathrm{G}_t = \overline{\mathrm{G}_t}$ for $t \leq n_0$ implies completeness for all t. We will also pay attention to the required degrees of the generators of the integral closure $\overline{\mathrm{G}}$ of the associated graded ring G.

2 Rees Algebras as Associated Graded Rings

In general, the Rees algebra of the conormal module $\mathfrak{p}/\mathfrak{p}^2$ and the associated graded ring of the prime ideal $\mathfrak{p}$ are distinct. They may even have different Krull dimensions. For example, if $(R, \mathfrak{m})$ is a local ring then the Rees algebra $\mathcal{R}(\mathfrak{m}/\mathfrak{m}^2)$ is a polynomial ring with $\nu(\mathfrak{m})$ number of variables over the residue field of R, where $\nu(\mathfrak{m})$ is the minimal number of generators of $\mathfrak{m}$. On the other hand, in the diagram illustrated at the beginning, we suggest that there may be a relationship between the associated graded ring of $\mathfrak{p}$ and the Rees algebra of $\mathfrak{p}/\mathfrak{p}^2$. We point out when this is possible.

Proposition 2.1 *Let $(R, \mathfrak{m})$ be a universally catenary Noetherian local ring, $\mathfrak{p}$ a prime ideal of R, and G the associated graded ring of $\mathfrak{p}$. There exists a surjection from G onto the Rees algebra $\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)$ of the conormal module $\mathfrak{p}/\mathfrak{p}^2$ if and only if $\mathfrak{p}$ is generically a complete intersection (i.e., $R_\mathfrak{p}$ is a regular local ring).*

Proof. If we assume that $\mathfrak{p}$ is generically a complete intersection, then we apply the dimension formula of [12, Lemma 1.2.2] to the Rees algebra $\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)$ to obtain the following.

$$\dim \mathcal{R}(\mathfrak{p}/\mathfrak{p}^2) \leq \dim R/\mathfrak{p} + \mathrm{edim}(R_\mathfrak{p}) = \dim R/\mathfrak{p} + \mathrm{height}(\mathfrak{p}) = \dim \mathrm{G},$$

where $\mathrm{edim}(R_\mathfrak{p})$ is the embedding dimension of $R_\mathfrak{p}$. This proves that $\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)$ is a homomorphic image of the associated graded ring G.

Now suppose that there is a surjective homomorphism from G to $\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)$, then

$$\mathrm{edim}(R_\mathfrak{p}) = \dim \mathcal{R}(\mathfrak{p}_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}^2) \leq \dim R_\mathfrak{p},$$

and hence $R_\mathfrak{p}$ is a regular local ring. $\square$

Suppose that a prime ideal $\mathfrak{p}$ is of linear type. This is equivalent to say that the natural map between the symmetric algebra $\mathcal{S}(\mathfrak{p}/\mathfrak{p}^2)$ and the associated graded ring G of $\mathfrak{p}$ is an isomorphism. The approximation complex $\mathcal{M}(\mathfrak{p})$ comes in at this point since zeroth Koszul homology module $H_0(\mathcal{M}(\mathfrak{p}))$ is $\mathcal{S}(\mathfrak{p}/\mathfrak{p}^2)$. One condition on the ideal $\mathfrak{p}$ that has an impact on the acyclicity of $\mathcal{M}(\mathfrak{p})$ is called G_∞: For each prime ideal $\mathfrak{q} \in V(\mathfrak{p})$, the minimal number of generators $\nu(\mathfrak{p}_\mathfrak{q})$ of $\mathfrak{p}_\mathfrak{q}$ is less than or equal to height of $\mathfrak{q}$. We are going to give a formulation of some conditions leading to associated graded rings which are domains. This formulation involves approximation complexes.

Proposition 2.2 *Let R be a Cohen–Macaulay local ring and $\mathfrak{p}$ a non-maximal prime ideal with sliding depth. Then $\mathfrak{p}$ satisfies G_∞ and the associated graded ring $\mathrm{G} = \bigoplus G_t$ of $\mathfrak{p}$ is a domain if and only if for every proper prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$,*

$$\nu(\mathfrak{p}_\mathfrak{q}) \leq \text{height}(\mathfrak{q}) - 1,$$

where $\nu(\mathfrak{p}_\mathfrak{q})$ is the minimal number of generators of $\mathfrak{p}_\mathfrak{q}$.

Proof. Suppose that $\nu(\mathfrak{p}_\mathfrak{q})$ is less than or equal to height$(\mathfrak{q}) - 1$ for every proper prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$. Then $\mathfrak{p}$ satisfies G_∞ and the approximation complex $\mathcal{M}(\mathfrak{p})$ is acyclic ([7, Theorem 5.1]). Let $S = \bigoplus S_t$ be the polynomial ring $R[T_1, \ldots, T_n]$ and H_i the ith Koszul homology module associated to $\mathfrak{p}$. Applying the Depth Lemma([12, Lemma 3.1.4]) to the approximation complex

$$0 \to H_r \otimes S_{t-r} \to \cdots \to H_0 \otimes S_t \to S_t(\mathfrak{p}/\mathfrak{p}^2) = G_t \to 0,$$

gives that each component G_t of G satisfies

$$\text{depth } (\mathrm{G}_t) > \text{depth } (H_r) - (r+1) \geq d - n + r - (r+1) \geq 0.$$

This implies that locally the algebra G is torsionfree as an $R/\mathfrak{p}$–module. If K is the field of fractions of $R/\mathfrak{p}$, we have an embedding $\mathrm{G} \hookrightarrow \mathrm{G} \otimes K$. Since $\mathrm{G} \otimes K$ is a ring of polynomials over K, the associated graded ring G is a domain.

For the converse, we may assume that $(R, \mathfrak{q})$ is a local ring. Since the ideal $\mathfrak{p}$ satisfies G_∞, the ideal $\mathfrak{p}$ is generated by a d–sequence and the minimal number of generators $\nu(\mathfrak{p})$ is equal to the analytic spread $\ell(\mathfrak{p})$ ([10, Theorem 2.2]). Therefore,

$$\nu(\mathfrak{p}) = \ell(\mathfrak{p}) = \dim \mathrm{G}/\mathfrak{q}\mathrm{G} < \dim \mathrm{G} = \text{height}(\mathfrak{q}),$$

where the inequality follows from the fact that G is a domain. $\square$

Proposition 2.3 *Let R be a Cohen–Macaulay local ring and $\mathfrak{p}$ a prime ideal with sliding depth. Suppose that $R/\mathfrak{p}$ is integrally closed. Then the associated graded ring* G *of $\mathfrak{p}$ is an integrally closed domain if for every proper prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$,*

$$\nu(\mathfrak{p}_\mathfrak{q}) \leq \max\{\text{height}(\mathfrak{p}_\mathfrak{q}) \ , \ \text{height}(\mathfrak{q}) - 2\}.$$

The converse holds if $\mathfrak{p}$ has finite projective dimension.

Proof. Let $\mathfrak{q}'$ be a prime ideal of G of height one and $\mathfrak{q}$ its inverse image in R. The associated graded ring G is Cohen-Macaulay ([7, Theorem 5.1]) and

$$\mathrm{G} = \bigcap \mathrm{G}_{\mathfrak{q}'}, \quad \text{for height}(\mathfrak{q}') = 1.$$

It is enough to show that height of $\mathfrak{q}' \cap R/\mathfrak{p}$ is one. Suppose that height$(\mathfrak{q}) - 2$ is greater than or equal to height of $\mathfrak{p}$. We may assume that $\mathfrak{q}$ is the maximal ideal of R. Then

$$\text{height}(\mathfrak{q}) - 2 \geq \nu(\mathfrak{p}) = \ell(\mathfrak{p}) = \dim(\mathrm{G}/\mathfrak{q}\mathrm{G}) = \dim \mathrm{G} - \text{height}(\mathfrak{q}\mathrm{G}).$$

Therefore,

$$\text{height}(\mathfrak{q}') \geq \text{height}(\mathfrak{q}\mathrm{G}) \geq 2.$$

For the converse, suppose that $\mathfrak{p}$ has finite projective dimension and that G is integrally closed. We make use of [11, Theorem 2.4], which asserts that the analytic spread $\ell(\mathfrak{p}_{\mathfrak{q}})$ is less than or equal to height$(\mathfrak{q}) - 2$, for the relevant primes of $V(\mathfrak{p})$. Since $\mathfrak{p}$ is of linear type, its analytic spread $\ell(\mathfrak{p}_{\mathfrak{q}})$ and minimum number of generators $\nu(\mathfrak{p}_{\mathfrak{q}})$ are the same. $\square$

Suppose that the associated graded ring G of $\mathfrak{p}$ is a domain but not integrally closed. We would like to describe the set of prime ideals $\mathfrak{q}$ of $R/\mathfrak{p}$ such that $\mathrm{G}_{\mathfrak{q}}$ is not integrally closed. The condition for the associated graded ring G to be a domain under the assumptions of Proposition 2.2 can be rephrased in terms of the Fitting ideals of the prime ideal $\mathfrak{p}$. Suppose that $\mathfrak{p}$ has a presentation

$$R^m \xrightarrow{\varphi} R^n \to \mathfrak{p} \to 0.$$

Denote the ideal generated by $t \times t$ minors of φ by $I_t(\varphi)$. For every prime ideal $\mathfrak{q}$ which contains $\mathfrak{p}$ properly, the condition

$$\nu(\mathfrak{p}_{\mathfrak{q}}) \leq \text{height}(\mathfrak{q}) - 1$$

is equivalent to

$$\text{grade } I_t(\varphi) \geq (n-1) - t + 3,$$

for $1 \leq t \leq n - \text{height}(\mathfrak{p})$ ([8, Corollary 6.7]). For an integer s such that height$(\mathfrak{p}) + 1 \leq s \leq \dim R$, letting $t = n - s + 1$ gives that

$$\text{height}(I_{n-s+1}(\varphi)) \geq s + 1.$$

Suppose that $R/\mathfrak{p}$ is integrally closed. We define the *normal locus* of the associated graded ring G as the set

$$\mathrm{NL}(\mathrm{G}) = \{\mathfrak{q} \in \mathrm{Spec}(R/\mathfrak{p}) \mid \mathrm{G}_{\mathfrak{q}} \text{ is an integrally closed domain}\}.$$

The following proposition shows that the normal locus of the associated graded ring G is determined by the Fitting ideals of $\mathfrak{p}$ or equivalently of $\mathfrak{p}/\mathfrak{p}^2$ (if $\mathfrak{p}$ has sliding depth).

Proposition 2.4 *Let R be a Cohen-Macaulay ring and $\mathfrak{p}$ a prime ideal with sliding depth. Suppose that $\mathfrak{p}$ satisfies G_∞, that $R/\mathfrak{p}$ is integrally closed and that the associated graded ring* G *of $\mathfrak{p}$ is a domain. Then the set* NL(G) *is an open subset of $V(\mathfrak{p})$.*

Proof. We may assume that G is not integrally closed. Suppose that $\mathfrak{p}$ has a presentation $R^m \xrightarrow{\varphi} R^n \to \mathfrak{p} \to 0$. Since G is a domain, using Proposition 2.2 and the argument above, we have

$$\text{height}(I_{n-s+1}(\varphi)) \geq s+1,$$

for all s such that $\text{height}(\mathfrak{p}) + 1 \leq s \leq n$.

On the other hand, since the associated graded ring G is not integrally closed, by Proposition 2.3, there exists a prime ideal $\mathfrak{q} \supset \mathfrak{p}$ of height $t \geq \text{height}(\mathfrak{p}) + 2$ such that $\nu(\mathfrak{p}_\mathfrak{q}) > \text{height}(\mathfrak{q}) - 2$. Because of the existence of $\mathfrak{q}$, $\text{height}(I_{n-t+2}(\varphi)) = t$. Now let $\Sigma = \{t \mid \text{height}(I_{n-t+2}(\varphi)) = t, \;\; \text{height}(\mathfrak{p}) + 2 \leq t \leq n\}$. For each $t \in \Sigma$, let $I_{n-t+2}(\varphi) = I'_{n-t+2} \bigcap I''_{n-t+2}$, where $\text{height}(I'_{n-t+2})$ is t and $\text{height}(I''_{n-t+2})$ is greater than t. Then the set NL(G) is the complement of $\bigcup_{t \in \Sigma} V(I'_{n-t+2})$. $\square$

Let us put this result under some perspective. Let S be a domain and E a finitely generated S–module. In general, we are not aware of a way to determine that the symmetric algebra of E is a domain using only the Fitting ideals of E. In contrast, this is possible for the conormal module with the properties above. The same observations apply to the obstructions to the normality of the symmetric algebra of E.

3 Divisor Class Groups

Throughout this section, we assume that $\mathfrak{p}$ is a prime R–ideal of finite projective dimension and that the associated graded ring G of $\mathfrak{p}$ is a domain. Under these assumptions the associated graded ring of $\mathfrak{p}$ is isomorphic to the Rees algebra of the conormal $R/\mathfrak{p}$–module $\mathfrak{p}/\mathfrak{p}^2$. We are going to compare the divisor class group of $R/\mathfrak{p}$ and that of G when they are both integrally closed domains. We recall that if R is an integrally closed domain, then $\text{Div}(R)$ is the group of divisorial ideals with the operation $I \circ J = ((IJ)^{-1})^{-1}$. The *divisor class group* $\mathsf{Cl}(R)$ is the quotient group $\text{Div}(R)/\text{Prin}(R)$, where $\text{Prin}(R)$ is the subgroup of principal fractional ideals ([5, Proposition 3.4 and §6]).

Theorem 3.1 *Let R be a Cohen–Macaulay ring and $\mathfrak{p}$ a prime ideal of finite projective dimension. If $R/\mathfrak{p}$ is an integrally closed domain of dimension greater than or equal to* 2 *and the associated graded ring* G *of $\mathfrak{p}$ is an integrally closed domain, then the mapping of divisor class groups*

$$\mathsf{Cl}(R/\mathfrak{p}) \ni [L] \mapsto [L\text{G}] \in \mathsf{Cl}(\text{G}),$$

is a group isomorphism. In particular, if G *is integrally closed and $R/\mathfrak{p}$ is a factorial domain, then* G *is a factorial domain.*

If $R/\mathfrak{p}$ has dimension less than or equal to one, then $\mathfrak{p}$ is a locally complete intersection. It follows that $\mathfrak{p}/\mathfrak{p}^2$ is projective and that the associated graded ring G of $\mathfrak{p}$ is isomorphic to the symmetric algebra $S(\mathfrak{p}/\mathfrak{p}^2)$, when the isomorphism between $\mathsf{Cl}(\text{G})$ and $\mathsf{Cl}(R/\mathfrak{p})$ will be taken care of by Proposition 3.3.

Our study of the divisor class group of an associated graded ring benefits enormously from a result of Johnson & Ulrich ([11]) that circumscribe very explicitly the Serre's condition (R_k) for the associated graded ring. In particular, we use [11, Theorem 2.4] in order to prove that the mapping in Theorem 3.1 is well defined.

Proposition 3.2 *Let R be a Cohen–Macaulay ring and $\mathfrak{p}$ a prime ideal of finite projective dimension. Suppose that the associated graded ring* G *of $\mathfrak{p}$ is an integrally closed domain. For $\mathfrak{q} \in \operatorname{Spec}(R/\mathfrak{p})$ of height at least 2, height of $\mathfrak{q}\mathrm{G}$ is greater than or equal to 2.*

Proof. Suppose that $\mathfrak{q}G$ is contained in a prime ideal $\mathfrak{P}$ of height 1. Setting $\mathfrak{m}$ to be the inverse image of $\mathfrak{P}$ in R and localizing, we may assume that $(R, \mathfrak{m})$ is a local ring and $\mathfrak{q}\mathrm{G} \subset \mathfrak{m}\mathrm{G} \subset \mathfrak{P}$. Now $\mathrm{height}(\mathfrak{m}\mathrm{G})$ is 1 and

$$\ell(\mathfrak{p}) = \dim \mathrm{G}/\mathfrak{m}\mathrm{G} = \dim \mathrm{G} - \mathrm{height}(\mathfrak{m}\mathrm{G}) = \dim R - 1,$$

which, by [11, Theorem 2.4], means that

$$\dim R - 1 = \ell(\mathfrak{p}) \leq \max\{\mathrm{height}(\mathfrak{p})\ ,\ \dim R - 2\} = \dim R - 2.$$

$\square$

Proposition 3.3 *Let R be a Krull domain and E a finitely generated projective R–module. Then the symmetric algebra $S = S_R(E)$ is a Krull domain and the mapping of divisor class groups*

$$\mathsf{Cl}(R) \longrightarrow \mathsf{Cl}(S), \quad [L] \mapsto [LS],$$

is a group isomorphism.

Proof. The proof is nearly the same as when the symmetric algebra S is a ring of polynomials. It consists of two main observations. Since the morphism $R \to S$ is flat, there is an induced homomorphism $\varphi : \mathsf{Cl}(R) \to \mathsf{Cl}(S)$ of divisor class groups ([5, Proposition 6.4]). We claim that φ is an isomorphism. Let I and J be two divisorial ideals of R. Note that

$$\mathrm{Hom}_S(I \otimes_R S, J \otimes_R S) = \mathrm{Hom}_R(I, J) \otimes_R S.$$

Let $\varphi = \bigoplus_{i\geq 0} \varphi_i \in \mathrm{Hom}_S(I \otimes_R S, J \otimes_R S)$ be the isomorphism between $I \otimes_R S$ and $J \otimes_R S$ with $\phi = \bigoplus_{i\geq 0} \phi_i$ such that $\varphi \circ \phi = \bigoplus_{i\geq 0} \varphi_i \circ \phi_i$ is the identity map. Since I and J are positively graded, it means that only $\varphi_0 \circ \phi_0$ is the identity map and the others must vanish. Therefore any isomorphism between $I \otimes_R S$ and $J \otimes_R S$ must be realized by an isomorphism between I and J, which shows that φ is injective.

In order to show that φ is surjective, it suffices to prove that for every divisorial prime ideal $\mathfrak{P}$ of S, its divisor class $[\mathfrak{P}]$ lies in the image of φ. Suppose that $\mathfrak{P} \cap R = \mathfrak{q} \neq 0$. Since $S_R(E)/\mathfrak{q}S_R(E) \simeq S_{R/\mathfrak{q}}(E/\mathfrak{q}E)$ which is a domain, $\mathfrak{q}S$ is a prime ideal so that $\mathfrak{P} = \mathfrak{q}S$. Suppose that $\mathfrak{q} = 0$. Let $0 \neq f \in \mathfrak{P}$ be an element that generates the extension of $\mathfrak{P}$ to KS, where K is the field of fractions of R. We recall that S is a projective R–module and that $f \in Q \subset S$, where Q is a direct summand of S. Let I be the R–ideal

$$I = (\ \alpha(f) \mid \alpha \in \mathrm{Hom}_R(Q, R)).$$

Then $\mathfrak{P} = I^{-1}fS$, which implies that $\varphi([I^{-1}]) = [I^{-1}S] = [\mathfrak{P}]$. $\square$

We are ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1: For any prime ideal of $R/\mathfrak{p}$ of the form $\mathfrak{q}/\mathfrak{p}$, of height at least 2, there is a regular sequence α, β in $\mathfrak{q}/\mathfrak{p}$, and therefore $(\mathfrak{q}/\mathfrak{p})[x]$ contains the prime element $f = \alpha + \beta x$. We claim that f is also a prime element in $\mathrm{G}[x]$. This follows from a series of observations. First, any minimal prime of (α, β) in $R/\mathfrak{p}$ has height 2 and therefore by Proposition 3.2, the extension $(\alpha, \beta)\mathrm{G}$ has height 2. Since G is integrally closed, the elements α and β generate an G–ideal of grade two and $f = \alpha + \beta x$ is a prime element of $\mathrm{G}[x]$ ([5, Lemma 14.1]).

Another elementary property of the calculation of divisor class groups is that it is unaffected when the rings are localized with respect to multiplicative sets formed by powers of prime elements. Let W be a multiplicative set generated by

$$\{\alpha + \beta x \mid \alpha, \beta \text{ is } R/\mathfrak{p}\text{–regular sequence}\}.$$

The elements $\alpha + \beta x$ in W are prime elements in both $(R/\mathfrak{p})[x]$ and $\mathrm{G}[x]$. Consider the following commutative diagram of divisor class groups.

$$\begin{array}{ccc}
\mathsf{Cl}(R/\mathfrak{p}) & \longrightarrow & \mathsf{Cl}(\mathrm{G}) \\
\cong\downarrow & & \downarrow\cong \\
\mathsf{Cl}((R/\mathfrak{p})[x]) & \longrightarrow & \mathsf{Cl}(\mathrm{G}[x]) \\
\cong\downarrow & & \downarrow\cong \\
\mathsf{Cl}((R/\mathfrak{p})[x]_W) & \longrightarrow & \mathsf{Cl}(\mathrm{G}[x]_W)
\end{array}$$

Denoting $(R/\mathfrak{p})[x]_W$ by A/J, the ideal J must be locally a complete intersection by [11, Theorem 2.4] and [3]. This means that J/J^2 is a finitely generated projective A/J–module and that $\mathrm{G}[x]_W$ is isomorphic to $S_{A/J}(J/J^2)$ ([8, Theorem 6.1]). By Proposition 3.3, we complete the proof. $\square$

Now we explore the homomorphism between divisor class groups in the case where $R/\mathfrak{p}$ is integrally closed but the associated graded ring G of $\mathfrak{p}$ is a domain that is not integrally closed. We assume that the integral closure $\overline{\mathrm{G}}$ of G is finitely generated as an G–module (a condition that holds true when R is an affine ring over a field, and in a greater generality).

Theorem 3.4 *Let R be a Cohen–Macaulay ring and $\mathfrak{p}$ a prime ideal of finite projective dimension. Suppose that $S = R/\mathfrak{p}$ is an integrally closed domain and that the associated graded ring* G *of $\mathfrak{p}$ is a domain with finite integral closure $\overline{\mathrm{G}}$. Then there exists an exact sequence of divisor class groups*

$$0 \to H \longrightarrow \mathsf{Cl}(\overline{\mathrm{G}}) \longrightarrow \mathsf{Cl}(R/\mathfrak{p}) \to 0,$$

where H is a finitely generated subgroup of $\mathsf{Cl}(\overline{\mathrm{G}})$ that vanishes if G *is integrally closed.*

Proof. We make two observations. First, for each prime ideal $\mathfrak{q} = Q/\mathfrak{p}$ of $S = R/\mathfrak{p}$ of height one, localizing at Q we get $QR_Q = (\mathfrak{p}, x)R_Q$, since $S_\mathfrak{q}$ is a discrete valuation ring. From the exact sequence

$$0 \to R_Q/\mathfrak{p}R_Q \xrightarrow{x} R_Q/\mathfrak{p}R_Q \longrightarrow R_Q/QR_Q \to 0,$$

it follows that the residue field of R_Q has finite projective dimension since $R/\mathfrak{p}$ has finite projective dimension by hypothesis. Therefore the local ring (R_Q, QR_Q) is a regular local ring by Serre's theorem [2, Theorem 2.2.7]. As a consequence $\mathfrak{p}R_Q$ is a complete intersection. We have thus shown that $\mathfrak{p}$ is a complete intersection in $\text{height}(\mathfrak{p}) + 1$.

Next, we may assume that the associated graded ring G is not integrally closed. Let $C = \text{ann}(\overline{\mathrm{G}}/\mathrm{G})$ be the conductor ideal. Since G and $\overline{\mathrm{G}}$ are graded rings, C is a homogeneous ideal of G. Set

$$L = C \cap S,$$

the component of C in degree zero. We claim that L is not contained in any prime ideal of S of height one. Suppose that L is contained in a prime ideal $\mathfrak{q} = Q/\mathfrak{p}$ of height one. Then we have $L_\mathfrak{q} \subset \mathfrak{q}S_\mathfrak{q}$. But $\mathfrak{p}R_Q$ is a complete intersection and therefore $\mathrm{G}_\mathfrak{q} = \text{gr}_{\mathfrak{p}R_Q} R_Q$ is a ring of polynomials over the discrete valuation ring $S_\mathfrak{q}$. This means that $\mathrm{G}_\mathfrak{q}$ is $\overline{\mathrm{G}}_\mathfrak{q}$ and $C_\mathfrak{q}$ is $S_\mathfrak{q}$.

By this observation, the S–ideal L has height greater than or equal to 2. Since S is integrally closed, there is a S-regular sequence a, b contained in L. Consider the addition of an indeterminate x to S, G and $\overline{\mathrm{G}}$. We consider the prime element $f = a + bx$ of $S[x]$. Since $f \in \text{ann}(\overline{\mathrm{G}}[x]/\mathrm{G}[x])$, we get $\overline{\mathrm{G}}[x]_f = \mathrm{G}[x]_f$. But $\mathrm{G}[x]_f$ is just the associated graded ring of the ideal $\mathfrak{p}R[x]_f$. This ideal inherits all the properties of $\mathfrak{p}$, in particular it is a prime ideal of finite projective dimension. Applying Theorem 3.1, we obtain these isomorphisms of divisor class groups

$$\mathsf{Cl}(\overline{\mathrm{G}}[x]_f) = \mathsf{Cl}(\mathrm{G}[x]_f) \cong \mathsf{Cl}(\text{gr}_{\mathfrak{p}R[x]_f} R[x]_f) \cong \mathsf{Cl}(R[x]_f/\mathfrak{p}R[x]_f) \cong \mathsf{Cl}(R/\mathfrak{p}).$$

In particular, the last isomorphism follows from the fact that f is a prime element of $S[x]$. On the other hand, using the exact sequence associated with the localization formula for divisor class groups [5, Corollary 7.2], we obtain

$$0 \to H \longrightarrow \mathsf{Cl}(\overline{\mathrm{G}}[x]) \longrightarrow \mathsf{Cl}(\overline{\mathrm{G}}[x]_f) \to 0,$$

where H is generated by the classes of all primes divisors in $\overline{\mathrm{G}}[x]$ that contain f so that H is finitely generated [5, Proposition 1.9]. Finally, replacing $\mathsf{Cl}(\overline{\mathrm{G}}[x]_f)$ with $\mathsf{Cl}(R/\mathfrak{p})$ and $\mathsf{Cl}(\overline{\mathrm{G}}[x])$ with $\mathsf{Cl}(\overline{\mathrm{G}})$ gives the desired sequence. □

While we do not know the structure of H in detail, examples and general arguments suggest that the following holds.

Conjecture 3.5 *H is a free group.*

Example 3.6 Let R be the polynomial ring $k[u, v, t, w]$, $\mathfrak{p}$ the R–ideal defining $S = R/\mathfrak{p} = k[x^3, x^2y, xy^2, y^3]$, and $\mathfrak{m}$ the S-ideal (x^3, x^2y, xy^2, y^3). Using Macaulay2, we compute the associated graded ring G and the integral closure $\overline{\mathrm{G}}$ of G.

$$\begin{aligned} \mathrm{G} &= R[x_1, x_2, x_3]/(\mathfrak{p}, -tx_1 + vx_2 - ux_3, wx_1 - tx_2 + vx_3) \\ \overline{\mathrm{G}} &= \mathrm{G}[Y]/(Yw - tx_3, Yt - vx_3, Yv - ux_3, Yu + vx_1 - ux_2, Y^2 - Yx_2 + x_1x_3). \end{aligned}$$

Then H is generated by one element $[\mathfrak{m}\overline{\mathrm{G}}]$ because

$$\overline{\mathrm{G}}/\mathfrak{m}\overline{\mathrm{G}} = k[x_1, x_2, x_3, Y]/(Y^2 - Yx_2 + x_1x_3),$$

where the monic quadratic polynomial is irreducible. We claim that $[\mathfrak{m}\overline{\mathrm{G}}]$ is a torsionfree element. Suppose that $[\mathfrak{m}^n\overline{\mathrm{G}}]$ is equal to $[f\overline{\mathrm{G}}]$ for some n, which means that $(\mathfrak{m}^n\overline{\mathrm{G}})^{-1-1} = f\overline{\mathrm{G}}$. Since $\mathfrak{m}^n\overline{\mathrm{G}}$ is a homogeneous ideal of height 1, f is not a unit and moreover has degree 0. Now $\mathfrak{m}^n$ is contained in $f\overline{\mathrm{G}}$ so that it is contained in fS, which is impossible because height$(\mathfrak{m}^n)$ is 2 but height(fS) is 1. This proves that H is isomorphic to $\mathbb{Z}$. Moreover, [5, Proposition 11.4] shows that $\mathsf{Cl}(S)$ is $\mathbb{Z}$. Therefore, $\mathsf{Cl}(\overline{\mathrm{G}})$ is $\mathbb{Z}\bigoplus\mathbb{Z}$.

Although we have concentrated on the Rees algebra $\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)$, there are related algebras to which the techniques employed here may be applied. For example, suppose that $R_\mathfrak{p}$ is a regular local ring and that the $\mathfrak{p}$–symbolic filtration $\{\mathfrak{p}^{(n)}, n \geq 0\}$ equals the filtration $\{\overline{\mathfrak{p}^n}, n \geq 0\}$. Then the reduced ring $\mathrm{G}_{\mathrm{red}}$ of the associated graded ring G is also a domain ([9, Theorem 2.1]). The significant difference between the algebras G and $\mathrm{G}_{\mathrm{red}}$ is that we have criteria for the (R_1)–condition for the associated graded ring G only, which is essential for the study of normality. Nevertheless we are able to obtain a similar result to Theorem 3.4 in case of the integral closure $\overline{\mathrm{G}_{\mathrm{red}}}$ of the reduced ring $\mathrm{G}_{\mathrm{red}}$.

Proposition 3.7 *Let R be a Cohen–Macaulay ring, $\mathfrak{p}$ a prime ideal of finite projective dimension, and G the associated graded ring of $\mathfrak{p}$. Suppose that $S = R/\mathfrak{p}$ is an integrally closed domain and that the reduced ring $\mathrm{G}_{\mathrm{red}}$ of G is a domain with finite integral closure $\overline{\mathrm{G}_{\mathrm{red}}}$. Then there exists an exact sequence of divisor class groups*

$$0 \to H \longrightarrow \mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}}) \longrightarrow \mathsf{Cl}(R/\mathfrak{p}) \to 0,$$

where H is a finitely generated subgroup of $\mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}})$.

Proof. If $\mathrm{G}_{\mathrm{red}}$ is integrally closed, then G is a domain ([9, Proposition 2.2]) so that $\mathrm{G}_{\mathrm{red}}$ is just G. Now we may assume that $\mathrm{G}_{\mathrm{red}}$ is not integrally closed. Let $C = \mathrm{ann}(\overline{\mathrm{G}_{\mathrm{red}}}/\mathrm{G}_{\mathrm{red}})$ be the conductor ideal. Then the the component of C in degree zero $L = C \cap S$ has grade at least two. We consider the prime element $f = a + bx$ of $S[x]$, where a, b is a S–regular sequence contained in L and x is an indeterminate.

Since $f \in \mathrm{ann}(\overline{\mathrm{G}_{\mathrm{red}}}[x]/\mathrm{G}_{\mathrm{red}}[x])$, we get $\overline{\mathrm{G}_{\mathrm{red}}}[x]_f = \mathrm{G}_{\mathrm{red}}[x]_f = \mathrm{G}[x]_f$, where the last equality is again from the result [9, Proposition 2.2]. Now we apply Theorem 3.1 and obtain the following isomorphisms of divisor class groups.

$$\mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}}[x]_f) = \mathsf{Cl}(\mathrm{G}[x]_f) \cong \mathsf{Cl}(\mathrm{gr}_{\mathfrak{p}R[x]_f} R[x]_f) \cong \mathsf{Cl}(R[x]_f/\mathfrak{p}R[x]_f) \cong \mathsf{Cl}(R/\mathfrak{p}).$$

By [5, Corollary 7.2], there is an exact sequence

$$0 \to H \longrightarrow \mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}}[x]) \longrightarrow \mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}}[x]_f) \to 0.$$

Finally, by replacing $\mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}}[x]_f)$ with $\mathsf{Cl}(R/\mathfrak{p})$ and $\mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}}[x])$ with $\mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}})$, we show that there is a surjective group homomorphism from $\mathsf{Cl}(\overline{\mathrm{G}_{\mathrm{red}}})$ to $\mathsf{Cl}(R/\mathfrak{p})$. $\square$

4 Integrally Closed Conormal Modules

If R is an integrally closed domain and E is a torsionfree R–module, the integral closure of the Rees algebra $\mathcal{R}(E) = \bigoplus_{n\geq 0} E_n$ of the module E is the algebra $\overline{\mathcal{R}(E)} = \bigoplus_{n\geq 0} \overline{E_n}$,

where $\overline{E_n}$ is the integral closure of the module E_n for all n (See Proposition 4.1 below). Partly for this reason, it is worthwhile to study the integral closure of a module and apply its techniques to the components of the associated graded ring. Throughout this section we assume that height of $\mathfrak{p}$ is at least two. If $\mathfrak{p}$ is a prime R–ideal which is generically a complete intersection and whose associated graded ring G is a domain, then G is isomorphic to the Rees algebra $\mathcal{R}(\mathfrak{p}/\mathfrak{p}^2)$ of the conormal module $\mathfrak{p}/\mathfrak{p}^2$. Under such assumptions, we examine the relationship between the completeness of the components of G and the normality of G.

Proposition 4.1 *Let R be a normal domain and E a torsionfree finitely generated R–module. The Rees algebra $\mathcal{R}(E) = \bigoplus_{n\geq 0} E_n$ of the module E is integrally closed if and only if E_n is integrally closed for all n.*

Proof. If each component E_n is integrally closed, we have

$$\mathcal{R}(E) = \sum_{n\geq 0} E_n = \sum_{n\geq 0}\bigcap_V VE_n = \bigcap_V V\mathcal{R}(E) = \bigcap_V \mathcal{R}(VE),$$

where V runs over all the valuation overrings of R. Since VE is a free V–module, the Rees algebra $\mathcal{R}(VE)$ is a ring of polynomials over V. This gives a representation of $\mathcal{R}(E)$ as an intersection of polynomial rings and it is thus normal. The converse is similar. □

Suppose that the associated graded ring $\mathrm{G} = \bigoplus_{t\geq 0} G_t$ satisfies the (S_2)–condition. Denote the integral closure of G by $\overline{\mathrm{G}}$ and the ideal $\bigoplus_{t\geq 1} G_t$ by G_+. Let I be the ideal generated by G–regular sequence of length two in G_+. Then depth $(I, \overline{\mathrm{G}}/\mathrm{G})$ is at least one and hence we can choose a $\overline{\mathrm{G}}/\mathrm{G}$–regular element x from the degree one component. Suppose that G_{t_0} is not integrally closed for some t_0. The map defined as the multiplication by x gives the embedding

$$\left(\overline{\mathrm{G}}/\mathrm{G}\right)_{t_0} \stackrel{\cdot x}{\longrightarrow} \left(\overline{\mathrm{G}}/\mathrm{G}\right)_{t_0+1},$$

and G_t is not integrally closed for every $t \geq t_0$. We shall be concerned with the following broad conjecture.

Conjecture 4.2 Let R be a Cohen–Macaulay ring and $\mathfrak{p}$ a prime ideal of finite projective dimension. Suppose that the associated graded ring G is a Cohen-Macaulay domain and that $R/\mathfrak{p}$ is integrally closed. If $\mathrm{G}_t = \mathfrak{p}^t/\mathfrak{p}^{t+1}$ is integrally closed for $t = \dim R/\mathfrak{p} - 1$, then G is normal.

If $R/\mathfrak{p}$ has dimension less than or equal to one, then the associated graded ring G is isomorphic to the symmetric algebra of the projective $R/\mathfrak{p}$–module $\mathfrak{p}/\mathfrak{p}^2$. Thus far we have settled the following case.

Theorem 4.3 *Let R be a Gorenstein local ring and $\mathfrak{p}$ a prime ideal generated by a strongly Cohen-Macaulay d–sequence. Suppose that $\mathfrak{p}$ has finite projective dimension, that $S = R/\mathfrak{p}$ is an integrally closed domain of dimension 2, and that the associated graded ring* G *of $\mathfrak{p}$ is a domain. Then the conormal module $E = \mathfrak{p}/\mathfrak{p}^2$ is integrally closed if and only if* G *is normal.*

Before we prove Theorem 4.3, we briefly recall how to attach divisors to certain modules (see [5]). Let R be a Noetherian normal domain and E a finitely generated torsionfree R–module of rank r. The *determinant* of E is rank one module $\det(E) = \wedge^r E$, while the *determinantal divisor* of E is the divisor class $\mathrm{div}(E)$ of the bidual $(\det(E))^{-1-1}$. The following are two elementary properties of this construction (we recall that the codimension $\mathrm{codim} A$ of a module A is the codimension of its annihilator). The first of these has an immediate proof.

Proposition 4.4 *Let R be a Noetherian normal domain and let*

$$0 \to A \longrightarrow B \longrightarrow C \longrightarrow D \to 0,$$

be an exact sequence of finitely generated R–modules. If $\mathrm{codim} A \geq 1$ *and* $\mathrm{codim} D \geq 2$, *then* $\mathrm{div}(B) = \mathrm{div}(C)$. *In particular, for any finitely generated torsionfree R–module E,* $\mathrm{div}(E) = \mathrm{div}(E^{-1-1})$.

Proposition 4.5 *Let R be a Noetherian normal domain. Suppose that the complex of finitely generated R–modules*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

is an exact sequence of free modules in each localization $R_{\mathfrak{p}}$ at height one prime ideals. Then $\mathrm{div}(B) = \mathrm{div}(A) + \mathrm{div}(C)$. *In particular, if B is a free R–module, then* $\mathrm{div}(A) = -\mathrm{div}(C)$.

Proof. We break up the complex into simpler exact complexes:

$$0 \to \ker(\varphi) \longrightarrow A \longrightarrow \mathrm{image}(\varphi) \to 0,$$

$$0 \to \mathrm{image}(\varphi) \longrightarrow \ker(\psi) \longrightarrow \ker(\psi)/\mathrm{image}(\varphi) \to 0,$$

$$0 \to \mathrm{image}(\psi) \longrightarrow C \longrightarrow C/\mathrm{image}(\psi) \to 0,$$

By hypothesis, $\mathrm{codim}\,\ker(\varphi) \geq 2$, $\mathrm{codim}\,\ker(\psi)/\mathrm{image}(\varphi) \geq 2$, $\mathrm{codim}\; C/\mathrm{image}(\psi) \geq 2$. By Proposition 4.4 we have the equality of determinantal divisors.

$$\mathrm{div}(A) = \mathrm{div}(\mathrm{image}(\varphi)) = \mathrm{div}(\ker(\psi)), \;\; \mathrm{div}(C) = \mathrm{div}(\mathrm{image}(\psi)).$$

What this all means is that we may assume the given complex is exact.

Suppose rank $(A) = r$ and rank $(C) = s$ and set $n = r + s$. Consider the pair $\wedge^r A$, $\wedge^s C$. For $v_1, \ldots, v_r \in A$, $u_1, \ldots, u_s \in C$, pick w_i in B such that $\psi(w_i) = u_i$ and consider

$$v_1 \wedge \cdots \wedge v_r \wedge w_1 \wedge \cdots \wedge w_s \in \wedge^n B.$$

Different choices for w_i would produce elements in $\wedge^n B$ that differ from the above by terms that would contain at least $r + 1$ factors of the form

$$v_1 \wedge \cdots \wedge v_r \wedge v_{r+1} \wedge \cdots,$$

with $v_i \in A$. Such products are torsion elements in $\wedge^n B$. This implies that modulo torsion we have a well defined pairing

$$[\wedge^r A/\text{torsion}] \otimes_R [\wedge^s C/\text{torsion}] \longrightarrow [\wedge^n B/\text{torsion}].$$

When localized at primes $\mathfrak{p}$ of codimension at most 1 the complex becomes an exact complex of projective $R_{\mathfrak{p}}$–modules and the pairing is an isomorphism. Upon taking biduals and the $\circ$ divisorial composition, we obtain the asserted isomorphism. $\square$

Proof of Theorem 4.3: Let n be the minimal number of generators of $\mathfrak{p}$. Since the associated graded ring G is a domain, n is either $\dim R - 2$ or $\dim R - 1$ (Proposition 2.2). Suppose that the associated graded ring G is not integrally closed and that the conormal module $E = \mathfrak{p}/\mathfrak{p}^2$ is integrally closed. If n is equal to $\dim R - 2$, then $\mathfrak{p}$ is generated by a regular sequence and so we may assume that n is equal to $\dim R - 1$. Since $R_{\mathfrak{p}}$ is a regular local ring, rank E is equal to height$(\mathfrak{p})$, that is $n-1$. Let

$$E = \overline{E} \hookrightarrow E_0 \hookrightarrow S^{n-1}$$

be the embedding of E into its bidual $E_0 = \mathrm{Hom}_S(\mathrm{Hom}_S(E, S), S)$.

At this point we need the notion of $\mathfrak{m}$–full modules. A torsionfree R–module E is called an $\mathfrak{m}$*–full module* if there is an element $x \in \mathfrak{m}$ such that $\mathfrak{m}E :_{R^r} x = E$. Integrally closed modules are $\mathfrak{m}$–full modules ([1, Proposition 2.6]) and since $\lambda(E_0/E)$ is finite, the minimal number $\nu(E_0)$ of generators of E_0 is either n or $n-1$ ([1, Corollary 2.7]).

Suppose $\nu(E_0)$ is equal to $n-1$. Consider the following diagram.

$$\begin{array}{ccccccc}
0 \to H_1 = \ker(\phi) & \longrightarrow & S^n & \xrightarrow{\phi} & E & \longrightarrow & 0 \\
\Big\downarrow{=} & & \Big\downarrow{=} & & \Big\downarrow{i} & & \\
0 \to \ker(\varphi) & \longrightarrow & S^n & \xrightarrow[\varphi]{} & E_0 = S^{n-1} & \longrightarrow & E_0/E \to 0
\end{array}$$

where the short exact sequence in the top row is from the approximation complex of E and i is an inclusion such that $\varphi = i \circ \phi$. In the exact sequence in the bottom row, by using Proposition 4.5, we get $\mathrm{div}(\ker(\varphi)) = \mathrm{div}(E_0/E)$, and hence

$$S \simeq \ker(\varphi) = \ker(\phi) = H_1.$$

This gives a contradiction because $\mathfrak{p}$ has finite projective dimension ([6, Theorem 1.4.9],[13, Theorem 5.2.1]).

Suppose $\nu(E_0)$ is equal to n. Consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & S^n & \xrightarrow{d_0} & E_0 & \longrightarrow & 0 \\
& & \Big\uparrow & & \Big\uparrow{A} & & \Big\uparrow{i} & & \\
0 & \longrightarrow & H_1 & \longrightarrow & S^n & \xrightarrow[d]{} & E & \longrightarrow & 0
\end{array}$$

We claim that K is isomorphic to the first Koszul homology module H_1 associated to $\mathfrak{p}$. Since $E_{\mathfrak{q}}$ is reflexive for every height 1 prime $\mathfrak{q}$, the determinantal divisor $\mathrm{div}(E)$ of E equals

$\operatorname{div}(E_0)$. By using Proposition 4.5, we obtain that $\operatorname{div}(K)$ equals $\operatorname{div}(H_1)$. Since K and H_1 are divisorial, the claim is proved. Let a, b be a S-regular sequence. By tensoring the exact sequence in the top row with $S/(a,b)S$, we have the short exact sequence

$$0 \to \overline{K} \to \overline{S^n} \to \overline{E_0} \to 0 \ ,$$

which splits because $\overline{H_1}$ is injective. Therefore, $\overline{E_0}$ is isomorphic to $\overline{S^{n-1}}$, which is a contradiction. □

Although this is far less than what we would wish, it will suffice to develop methods to set up the computation of the integral closure of some associated graded domains. Under the conditions of the theorem, we actually prove the following corollary.

Corollary 4.6 *If $S = R/\mathfrak{p}$ is an integrally closed domain, of arbitrary dimension, then the conormal module $E = \mathfrak{p}/\mathfrak{p}^2$ is integrally closed if and only if it is reflexive.*

Proof. Consider the following approximation complex

$$0 \to H_1 \to S^n \to E \to 0.$$

Using the Depth Lemma([12, Lemma 3.1.4]),

$$\text{depth } (E) \geq \min\{\text{depth } (H_1) - 1, \text{depth } (S^n)\} = \dim S - 1.$$

Now it is enough to consider prime S–ideals whose height is less than or equal to two. For every prime ideal $\mathfrak{q}$ of height one, $E_{\mathfrak{q}}$ is free because it is a finitely generated torsionfree module over a discrete valuation ring. For prime ideals $\mathfrak{q}$ of height two, Theorem 4.3 shows that $E_{\mathfrak{q}}$ is free. Therefore E satisfies the (S_2)–condition. □

Now we give an independent proof of Theorem 4.3 when the prime ideal $\mathfrak{p}$ has height 2. It indicates where we should look the integral closure of the conormal module. It is a consequence of very useful criterion of completeness.

Proposition 4.7 *Let R be a Cohen–Macaulay local ring and $\mathfrak{p}$ a prime R–ideal of height 2 with a finite free resolution*

$$0 \to R^2 \xrightarrow{\phi} R^3 \to \mathfrak{p} \to 0.$$

Suppose that $S = R/\mathfrak{p}$ is integrally closed. Then the determinant $\det(E)$ *of the conormal module $E = \mathfrak{p}/\mathfrak{p}^2$ is divisorial.*

Proof. The conormal module E has rank 2 with an embedding $E \hookrightarrow S^2$. Composing the embedding with the free resolution of E, we have the following exact sequence

$$S^2 \xrightarrow{\overline{\phi}} S^3 \xrightarrow{\varphi} S^2,$$

$$\phi = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}, \text{ and } \varphi = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix},$$

where the image of φ is E. The ideal $J = (a,b,c)$ of R contains $\mathfrak{p}$ properly so that J is a complete intersection. Since the S–ideal $J' = J/\mathfrak{p}$ has no embedded primes, J' is divisorial. Using $0 = \varphi \circ \overline{\phi}$, we have

$$\frac{b_1c_2 - b_2c_1}{\overline{a}} = \frac{a_2c_1 - a_1c_2}{\overline{b}} = \frac{a_1b_2 - a_2b_1}{\overline{c}}.$$

Then $\det(E) = I_2(\varphi) = \delta J'$, which proves that $\det(E)$ is divisorial. $\square$

Proposition 4.8 *Let S be an integrally closed domain and E a finitely generated integrally closed torsionfree S-module whose order determinant* $\det(E)$ *is divisorial. Then E is reflexive.*

Proof. Let $E_0 = \mathrm{Hom}_S(\mathrm{Hom}_S(E,S),S)$ and $x \in E_0 \setminus E$. Since $(E_0)_\mathfrak{p} = E_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of height one, the ideal $L = E : x$ has height greater than one. Choose elements $e_1, \ldots, e_{r-1}$ from E. Then

$$L(x \wedge e_1 \wedge \ldots \wedge e_{r-1}) = Lx \wedge e_1 \wedge \ldots \wedge e_{r-1} \subset \det(E)$$

Since $\mathrm{height}(L) \geq 2$ and $\det(E)$ is divisorial, $x \wedge e_1 \wedge \ldots \wedge e_{r-1} \in \det(E)$, i.e., $\det(x,E) = \det(E)$. Recall that if $\det(F) = \det(E)$ for $F \supset E$, then $F \subseteq \overline{E}$. Now (x,E) is integral over E, which means that $x \in \overline{E} = E$. $\square$

Example 4.9 Let R be the polynomial ring $k[u,v,t,w]$ and $\mathfrak{p}$ the R-ideal generated by $(uw - tv, ut - v^2, vw - t^2)$. Note that $S = R/\mathfrak{p}$ is isomorphic to $k[x^3, x^2y, xy^2, y^3]$. Let E be the conormal module $\mathfrak{p}/\mathfrak{p}^2$, E_0 its bidual $\mathrm{Hom}_S(\mathrm{Hom}_S(E,S),S)$. It is easy to check that the ideal $\mathfrak{p}$ satisfies the assumptions of Theorem 4.3. In particular the R-ideal $\mathfrak{p}$ and the S-module E have the following presentations

$$0 \longrightarrow R^2 \stackrel{\phi}{\longrightarrow} R^3 \longrightarrow \mathfrak{p} \longrightarrow 0, \quad \phi = \begin{bmatrix} -t & w \\ v & -t \\ -u & v \end{bmatrix},$$

$$S^2 \stackrel{\overline{\phi}}{\longrightarrow} S^3 \stackrel{\varphi}{\longrightarrow} S^2, \qquad \varphi = \begin{bmatrix} w & 0 & -t \\ 0 & u & -v \end{bmatrix},$$

where the image of φ is E. Using Macaulay2, we compute the associated graded ring G and its integral closure $\overline{\mathrm{G}}$.

$$\begin{aligned} \mathrm{G} &= R[x_1,x_2,x_3]/(\mathfrak{p}, -tx_1 + vx_2 - ux_3, wx_1 - tx_2 + vx_3), \\ \overline{\mathrm{G}} &= \mathrm{G}[Y]/(Yw - tx_3, Yt - vx_3, Yv - ux_3, Yu + vx_1 - ux_2, Y^2 - Yx_2 + x_1x_3). \end{aligned}$$

Since the module E is not reflexive, E is not integrally closed (Corollary 4.6). Having $\det(E) = \det(E_0) = v(t,w,u)$ gives that the integral closure $\overline{E}$ is the bidual module E_0. Furthermore we claim that $\overline{\mathrm{G}}$ is Cohen–Macaulay. Let L be $\mathrm{G}[Y]$–ideal generated by $(Yw -$

$tx_3, Yt - vx_3, Yv - ux_3, Yu + vx_1 - ux_2)$. Note that $Y^2 - Yx_2 + x_1x_3$ is a monic irreducible polynomial. From the exact sequence

$$0 \to L \to \mathrm{G}[Y]/(Y^2 - Yx_2 + x_1x_3) \to \overline{\mathrm{G}} \to 0,$$

we obtain

$$\mathrm{Hom}_{\mathrm{G}}(\overline{\mathrm{G}}, \mathrm{G}) = (\mathrm{G} :_{\mathrm{G}} \overline{\mathrm{G}}) \simeq (\beta \mid \alpha + \beta Y \in L) = \mathfrak{m}\mathrm{G}.$$

Since G is Gorenstein, by [12, Proposition 1.1.15], $\overline{\mathrm{G}}$ is reflexive, and hence

$$\overline{\mathrm{G}} = \mathrm{Hom}_{\mathrm{G}}(\mathrm{Hom}_{\mathrm{G}}(\overline{\mathrm{G}}, \mathrm{G}), \mathrm{G}) \simeq \mathrm{Hom}_{\mathrm{G}}(\mathfrak{m}\mathrm{G}, \mathrm{G}).$$

Since $\mathfrak{m}\mathrm{G}$ is Cohen–Macaulay, the integral closure $\overline{\mathrm{G}}$ is Cohen–Macaulay.

We are going to study the module theoretic properties of some components of G. It is helpful to find the integral closure in some cases.

Proposition 4.10 *Let R be a Gorenstein local ring, $\mathfrak{p}$ a perfect prime ideal generated by a strongly Cohen–Macaulay d–sequence, and $\mathrm{G} = \bigoplus_{i \geq 0} \mathrm{G}_i$ the associated graded ring of $\mathfrak{p}$. Suppose that $S = R/\mathfrak{p}$ is an integrally closed domain and that for every proper prime ideal $\mathfrak{q} \supset \mathfrak{p}$,*

$$\nu(\mathfrak{p}_{\mathfrak{q}}) \leq \mathrm{height}(\mathfrak{q}) - 1,$$

where $\nu(\mathfrak{p}_{\mathfrak{q}})$ is the minimal number of generators of $\mathfrak{p}_{\mathfrak{q}}$. Denote the minimal number of generators of $\mathfrak{p}$ by n and height of $\mathfrak{p}$ by g. Then G_{n-g} is not reflexive.

Proof. Suppose that G_{n-g} is reflexive. By Proposition 2.2, the associated graded ring G is a domain. By applying the Depth lemma([12, Lemma 3.1.4]) to the approximation complex

$$0 \to H_{n-g} \xrightarrow{\phi} H_{n-g-1} \otimes S_1 \to \cdots \to H_0 \otimes S_{n-g} \to \mathrm{G}_{n-g} \to 0\ ,$$

we show that the cokernel of ϕ, denoted by L, is a maximal Cohen–Macaulay module. Because the $(n-g)$th Koszul homology module H_{n-g} is the canonical module of S, the exact sequence $0 \to H_{n-g} \xrightarrow{\phi} H_{n-g-1} \otimes S_1 \to L \to 0$ splits. On the other hand, we have the following diagram of equivalences.

$$\begin{array}{ccc}
\mathrm{Hom}_S(H_{n-g} \bigoplus L, H_{n-g}) & = & \mathrm{Hom}_S(H_{n-g-1} \otimes S_1, H_{n-g}) \\
\| & & \| \\
\mathrm{Hom}_S(H_{n-g}, H_{n-g}) \bigoplus \mathrm{Hom}_S(L, H_{n-g}) & = & \bigoplus^n \mathrm{Hom}_S(H_{n-g-1}, H_{n-g}) \\
\| & & \| \\
S \bigoplus \mathrm{Hom}_S(L, H_{n-g}) & = & \bigoplus^n H_1
\end{array}$$

This contradicts that H_1 cannot have S as a direct summand ([6, Theorem 1.4.9], [13, Theorem 5.2.1]). $\square$

Suppose that G is not integrally closed. Under the same assumptions as those in Theorem 4.3, there are examples which show that G_{d-g-1} is not necessarily the first component which is not integrally closed. On the other hand, if G_i is integrally closed for all $i < d-g-1$, then we may have a better understanding of the difference between G_{d-g-1} and its bidual $\mathrm{G}^{**}_{d-g-1} = \mathrm{Hom}_S(\mathrm{Hom}_S(\mathrm{G}_{d-g-1}, S), S)$.

Proposition 4.11 *Let R be a Gorenstein local ring of dimension $d \geq 3$, $\mathfrak{p}$ a perfect prime ideal of height 2, generated by a d–sequence. Let φ be the matrix of syzygies of $\mathfrak{p}$. Suppose that $S = R/\mathfrak{p}$ is an integrally closed domain and that for every proper prime ideal $\mathfrak{q} \supset \mathfrak{p}$,*

$$\nu(\mathfrak{p}_\mathfrak{q}) \leq \mathrm{height}(\mathfrak{q}) - 1,$$

where $\nu(\mathfrak{p}_\mathfrak{q})$ is the minimal number of generators of $\mathfrak{p}_\mathfrak{q}$. Then

$$\mathrm{G}^{**}_{n-2}/\mathrm{G}_{n-2} \simeq \mathrm{Ext}^d_R(R/I_1(\varphi), R),$$

where $\mathrm{G}_{n-2} = \mathfrak{p}^{n-2}/\mathfrak{p}^{n-1}$ and $\mathrm{G}^{**}_{n-2} = \mathrm{Hom}_S(\mathrm{Hom}_S(\mathrm{G}_{n-2}, S), S)$.

Proof. By Proposition 2.2, the associated graded ring $\mathrm{G} = \bigoplus_{t\geq 0} \mathrm{G}_t$ of the prime ideal $\mathfrak{p}$ is a domain. We denote $\mathrm{G}^{**}_{n-2}/\mathrm{G}_{n-2}$ by C. From the following two exact sequences

$$0 \to H_{n-2} \xrightarrow{\phi} H_{n-3} \otimes S_1 \to \cdots \to H_1 \otimes S_{n-3} \to S_{n-2} \to \mathrm{G}_{n-2} \to 0,$$

$$0 \to \mathrm{G}_{n-2} \to \mathrm{G}^{**}_{n-2} \to C \to 0,$$

we obtain that

$$\mathrm{Ext}^{d-2}_S(C, H_{n-2}) \simeq \mathrm{coker}\ \xi,$$

where $\xi : \mathrm{Hom}_S(H_{n-3} \otimes S_1, H_{n-2}) \to \mathrm{Hom}_S(H_{n-2}, H_{n-2})$ is the dual map of ϕ. We claim that

$$\mathrm{coker}\ \xi \simeq R/I_1(\varphi)\ .$$

Set the matrix of syzygies $\varphi = [a_{ij}] = [v_1\ \ldots\ v_{n-1}]$, where $1 \leq i \leq n$, $1 \leq j \leq n-1$ and v_s's are the column vectors. For each i, we have

$$\sum_{j=1}^{n-1} (-1)^{j-1} a_{ij} \bigwedge_{s=1\ ,\ s\neq j}^{n-1} v_s \ \in\ \mathfrak{p} \bigwedge^{n-2} R^n.$$

For each $j = 1, \ldots, n-1$, the map ϕ is defined in the following manner.

$$\begin{array}{rcl} \phi : \overline{\bigwedge_{s=1\ ,\ s\neq j}^{n-1} v_s} & \mapsto & \sum_{t=1\ ,\ t\neq j}^{n-1} (-1)^t \overline{\bigwedge_{s=1\ ,\ s\neq j,t}^{n-1} v_s} \otimes v_t \\ & & \| \\ & & \sum_{i=1}^{n} \left(\sum_{t=1\ ,\ t\neq j}^{n-1} (-1)^t \overline{a_{it} \bigwedge_{s=1\ ,\ s\neq j,t}^{n-1} v_s} \right) e_i. \end{array}$$

For any $h = \sum_{i=1}^n h_i \epsilon_i \in \mathrm{Hom}(H_{n-3} \otimes S_1, H_{n-2})$, the map ξ is defined as the following.

$$\xi(h) = h \circ \phi : \overline{\bigwedge_{s=1\ ,\ s\neq j}^{n-1} v_s} \mapsto \sum_{i=1}^{n} \left(\sum_{t=1\ ,\ t\neq j}^{n-1} (-1)^t \overline{a_{it} \bigwedge_{s=1\ ,\ s\neq j,t}^{n-1} v_s \bigwedge h_i} \right).$$

Let h_{ij} be $v_j\epsilon_i$, where $1 \le i \le n$ and $1 \le j \le n-1$. For example, in case when $j=1$,

$$\xi(h_{i1})\left(\overline{\bigwedge_{s=2}^{n-1} v_s}\right) = \sum_{t=2}^{n-1}(-1)^t a_{it}\overline{\bigwedge_{s=1\,,\,s\neq t}^{n-1} v_s} = a_{i1}\overline{\bigwedge_{s=2}^{n-1}} v_s.$$

Similarly we show that for each $1 \le i \le n$ and $1 \le j \le n-1$, the map $\xi(h_{ij})$ is the multiplication by a_{ij}, which proves the claim. Let $\underline{a} : a_1, a_2$ be an R-sequence in $\mathfrak{p}$ and $\underline{b} : b_1, \ldots, b_{d-2}$ in R such that $\underline{a}, \underline{b}$ is a system of parameters of R. Let J' be the S-ideal generated by the images of b_i's in S for all $i = 1, \ldots, d-2$. Finally we obtain the following natural isomorphisms:

$$\begin{aligned} C &\simeq \mathrm{Ext}_S^{d-2}(\mathrm{Ext}_S^{d-2}(C, H_{n-2}), H_{n-2}) \\ &\simeq \mathrm{Ext}_S^{d-2}(\mathrm{coker}\ \xi, H_{n-2}) \\ &\simeq \mathrm{Hom}_{S/J'}(\mathrm{coker}\ \xi, \omega_S/J'\omega_S) \\ &\simeq \mathrm{Hom}_{S/J'}(\mathrm{coker}\ \xi, \omega_{S/J'}) \\ &\simeq \mathrm{Hom}_{R/(\underline{a},\underline{b})}(\mathrm{coker}\ \xi, R/(\underline{a},\underline{b})) \\ &\simeq \mathrm{Ext}_R^d(R/I_1(\varphi), R). \end{aligned}$$

□

Corollary 4.12 *Let R be a Gorenstein local ring, $\mathfrak{p}$ a perfect prime ideal generated by a strongly Cohen–Macaulay d–sequence and E the conormal module $\mathfrak{p}/\mathfrak{p}^2$. Suppose that $S = R/\mathfrak{p}$ is an integrally closed domain of dimension 2, that $\nu(\mathfrak{p})$ is 3, and that* height$(\mathfrak{p})$ *is 2. Then $\nu(\overline{E})$ equals to 4.*

Example 4.13 Let $R = k[x, y, z, u, v]$ and let A be the matrix of syzygies of R–ideal $\mathfrak{p}$:

$$A = \begin{bmatrix} x & v-u & z \\ y & x & v \\ z & u & x \\ u & z & y \end{bmatrix}$$

The associated graded ring $\mathrm{G} = \mathrm{gr}_{\mathfrak{p}}R = \bigoplus \mathrm{G}_i$ is a domain but it is not integrally closed. The conormal module $E = \mathrm{G}_1 = \mathfrak{p}/\mathfrak{p}^2$ is reflexive. But $\nu(\mathrm{G}_2)$ and $\nu(\mathrm{G}_2^{**})$ are 10 and 11 respectively. This shows that G_2 is not integrally closed because of the failure of the $\mathfrak{m}$-fullness. In particular, by Proposition 4.11, G_2^{**} is the integral closure of G_2.

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